

On Chapline-Manton couplings: a cohomological approach

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Abstract

Chern-Simons couplings between Yang-Mills gauge fields and an abelian 2-form are derived by means of cohomological arguments.

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1 Introduction

The problem of constructing consistent interactions among fields with gauge freedom [1]–[4] has been reformulated [5] in the framework of the antifield BRST formalism [6]–[10] as a problem of consistent deformation of the master equation in the sense of deformation theory [11]. This technique has been applied to Chern-Simons models [5], Yang-Mills theories [12], and two-form gauge fields [13].

In this paper we investigate another interesting interaction, namely, the consistent interaction between the Yang-Mills vector potential and an abelian 2-form. As it will be seen, the procedure to be developed leads to combined Yang-Mills-2-form system coupled through a Yang-Mills Chern-Simons term

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(the Chapline-Manton model) [14]–[17]. Chern-Simons couplings of a two-form to Yang-Mills gauge fields play a crucial role in the Green-Schwarz anomaly cancellation mechanism [18] and, therefore, are important in string theory [19].

The strategy employed in this paper is the following. We start with an action describing pure Yang-Mills theory and an abelian 2-form (the “free” theory), and determine the BRST differential s of this uncoupled model, which can be written as the sum between the Koszul-Tate differential and the exterior longitudinal derivative along the gauge orbits, $s = \delta + \gamma$. Next, we deform the solution to the master equation of the uncoupled model. The first-order deformation belongs to $H^0(s|d)$, where d is the exterior space-time derivative. As $s = \delta + \gamma$, the computation of the cohomology $H^0(s|d)$ proceeds by expanding the co-cycles according to the antighost number. Subsequently, we derive the second-order deformation, and thus, the solution to the master equation of the interacting theory (the higher-order deformation terms vanish). The antighost number zero piece in the deformed solution is nothing but the action of the Chapline-Manton model, while the terms linear in the antifields of the original fields and ghosts indicate that the added interactions deform the gauge transformations, as well as the gauge algebra. In this way, our approach emphasizes a complex consistent interaction that deforms both the gauge transformations and their algebra.

2 BRST symmetry for the “free” theory

In this section we derive the BRST symmetry for the “free” theory. In this respect, we begin with a Lagrangian action equal with the sum between the actions of Yang-Mills theory and a free 2-form

$$S_0^L[A_\mu^a, B_{\mu\nu}] = \int d^D x \left(-\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - \frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} \right), \quad (1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f_{bc}^a A_\mu^b A_\nu^c, \quad (2)$$

$$F_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} \equiv \partial_{[\mu} B_{\nu\rho]}. \quad (3)$$

Action (1) is invariant under the gauge transformations

$$\delta_\epsilon A_\mu^a = (D_\mu)_b^a \epsilon^b, \quad \delta_\epsilon B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \equiv \partial_{[\mu} \epsilon_{\nu]}, \quad (4)$$

with $(D_\mu)^a_b = \delta^a_b \partial_\mu + f^a_{bc} A_\mu^c$ the covariant derivative. The gauge transformations of the 2-form are first-stage reducible as they vanish for $\epsilon_\mu = \partial_\mu \epsilon$. Consequently, the solution to the master equation of the “free” theory reads as

$$S = S_0^L [A_\mu^a, B_{\mu\nu}] + \int d^D x \left(A_a^{*\mu} (D_\mu)^a_b \eta^b - \frac{1}{2} f^a_{bc} \eta_a^* \eta^b \eta^c + B^{*\mu\nu} \partial_{[\mu} \eta_{\nu]} + \eta^{*\mu} \partial_\mu \eta \right). \quad (5)$$

In (5), η^a and η_μ denote the fermionic ghosts with ghost number one, while η represents the bosonic ghost for ghost of ghost number two that appears due to the reducibility. The variables $A_a^{*\mu}$, $B^{*\mu\nu}$, η_a^* , $\eta^{*\mu}$ and η^* stand for the antifields. The first two sets of antifields have ghost number minus one, the next two sets display ghost number minus two, and the last possesses ghost number minus three. The ghost number is defined like the difference between pure ghost number (*pg*) and antighost number (*antigh*), where

$$pg(A_\mu^a) = 0, \quad pg(B_{\mu\nu}) = 0, \quad pg(A_a^{*\mu}) = 0, \quad (6)$$

$$pg(B^{*\mu\nu}) = 0, \quad pg(\eta_a^*) = 0, \quad pg(\eta^{*\mu}) = 0, \quad pg(\eta^*) = 0, \quad (7)$$

$$pg(\eta^a) = 1, \quad pg(\eta_\mu) = 1, \quad pg(\eta) = 2, \quad (8)$$

$$antigh(A_\mu^a) = 0, \quad antigh(B_{\mu\nu}) = 0, \quad antigh(A_a^{*\mu}) = 1, \quad (9)$$

$$antigh(B^{*\mu\nu}) = 1, \quad antigh(\eta_a^*) = 2, \quad antigh(\eta^{*\mu}) = 2, \quad (10)$$

$$antigh(\eta^*) = 3, \quad antigh(\eta^a) = 0, \quad antigh(\eta_\mu) = 0, \quad antigh(\eta) = 0. \quad (11)$$

The BRST differential $s_\bullet = (\bullet, S)$ of the “free” theory splits as

$$s = \delta + \gamma, \quad (12)$$

where δ is the Koszul-Tate differential and γ represents the longitudinal exterior derivative along the gauge orbits. The symbol $(,)$ is the usual notation for antibracket in the antifield formalism. Thus, we have

$$\delta A_\mu^a = 0, \quad \gamma A_\mu^a = (D_\mu)^a_b \eta^b, \quad (13)$$

$$\delta B_{\mu\nu} = 0, \quad \gamma B_{\mu\nu} = \partial_{[\mu} \eta_{\nu]}, \quad (14)$$

$$\delta\eta^a = 0, \quad \gamma\eta^a = -\frac{1}{2}f^a_{bc}\eta^b\eta^c, \quad (15)$$

$$\delta\eta_\mu = 0, \quad \gamma\eta_\mu = \partial_\mu\eta, \quad (16)$$

$$\delta\eta = 0, \quad \gamma\eta = 0, \quad (17)$$

$$\delta A_a^{*\mu} = -(D_\nu)_a^b F_b^{\nu\mu}, \quad \gamma A_a^{*\mu} = f^b_{ac} A_b^{*\mu} \eta^c, \quad (18)$$

$$\delta B^{*\mu\nu} = -\frac{1}{2}\partial_\rho F^{\rho\mu\nu}, \quad \gamma B^{*\mu\nu} = 0, \quad (19)$$

$$\delta\eta_a^* = (D_\mu)_a^b A_b^{*\mu}, \quad \gamma\eta_a^* = -f^b_{ac}\eta_b^* \eta^c, \quad (20)$$

$$\delta\eta^{*\mu} = -2\partial_\nu B^{*\nu\mu}, \quad \gamma\eta^{*\mu} = 0, \quad (21)$$

$$\delta\eta^* = \partial_\mu\eta^{*\mu}, \quad \gamma\eta^* = 0, \quad (22)$$

where $(D_\mu)_a^b = \delta_a^b \partial_\mu - f^b_{ac} A_\mu^c$. Formulas (13–22) will be useful in the next section at deforming the solution (5) by means of cohomological arguments.

3 Deformed solution of the master equation

Here, we will use the technique of deformation of the master equation with respect to the “free” theory, and will deduce the consistent interactions that can be introduced between the Yang-Mills vector potential and the 2-form.

A consistent deformation of action (1) and of its gauge invariances defines a deformation of the corresponding solution to the master equation that preserves both the master equation and field/antifield spectra. So, if

$$S_0^L [A_\mu^a, B_{\mu\nu}] + g \int d^D x \alpha_0 + g^2 \int d^D x \beta_0 + O(g^3), \quad (23)$$

stands for a consistent deformation of action (1), with deformed gauge transformations

$$\bar{\delta}_\epsilon A_\mu^a = (D_\mu)_a^b \epsilon^b + g\sigma_\mu^a + O(g^2), \quad (24)$$

$$\bar{\delta}_\epsilon B_{\mu\nu} = \partial_{[\mu} \epsilon_{\nu]} + g\xi_{\mu\nu} + O(g^2), \quad (25)$$

then the deformed solution to the master equation

$$\bar{S} = S + g \int d^D x \alpha + g^2 \int d^D x \beta + O(g^3) = S + gS_1 + g^2S_2 + O(g^3), \quad (26)$$

satisfies

$$(\bar{S}, \bar{S}) = 0, \quad (27)$$

where

$$\alpha = \alpha_0 + A_a^{*\mu} \bar{\sigma}_\mu^a + B^{*\mu\nu} \bar{\xi}_{\mu\nu} + \text{“more”}. \quad (28)$$

The master equation (27) splits according to the deformation parameter g as

$$s\alpha = \partial_\mu k^\mu, \quad (29)$$

$$s\beta + \frac{1}{2}\omega = \partial_\mu \theta^\mu, \quad (30)$$

\vdots

for some local k^μ and θ^μ , with

$$(S_1, S_1) = \int d^D x \omega. \quad (31)$$

Obviously, equation (27) is automatically satisfied at order zero in the coupling constant, and has thus been omitted. Equation (29) shows that the non-trivial first-order consistent interactions belong to $H^0(s|d)$, where d is the exterior space-time derivative. In the case where α is a coboundary modulo d ($\alpha = s\lambda + \partial_\mu \pi^\mu$), then the deformation is trivial (it can be eliminated by a redefinition of the fields).

In order to solve equation (29) we expand α accordingly the antighost number

$$\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_k, \quad \text{antigh}(\alpha_i) = i, \quad (32)$$

where the last term in (32) can be assumed to be annihilated by γ . Thus, since $\text{antigh}(\alpha_k) = k$ and $gh(\alpha_k) = 0$, the pure ghost number of α_k is equal to k . The fact that

$$\rho = \frac{1}{3} f_{abc} \eta^a \eta^b \eta^c, \quad (33)$$

is γ -invariant enforces that $k = 3m > 0$. Then,

$$\alpha_k = \alpha_{3m} = \mu_{3m}(\rho)^m, \quad (34)$$

where μ_{3m} belongs to $H_{3m}(\delta|d)$. As in the case of the theory under study $H_j(\delta|d) = 0$ for $j > 3$ [20], it follows that the last term in (32) has the form

$$\alpha_3 = \frac{1}{3} \mu_3 f_{abc} \eta^a \eta^b \eta^c, \quad (35)$$

with μ_3 from $H_3(\delta|d)$, hence solution to the equation

$$\delta\mu_3 + \partial_\mu j^\mu = 0, \quad (36)$$

for some j^μ . From the former relation in (22) we find that the general representative of $H_3(\delta|d)$ is

$$\mu_3 = \eta^*, \quad (37)$$

therefore

$$\alpha_3 = \frac{1}{3} f_{abc} \eta^* \eta^a \eta^b \eta^c. \quad (38)$$

At antighost number two, equation (29) takes the form

$$\delta\alpha_3 + \gamma\alpha_2 = \partial_\mu v^\mu, \quad (39)$$

for some v^μ . As

$$\delta\alpha_3 = -\frac{1}{3} f_{abc} (\partial_\mu \eta^{*\mu}) \eta^a \eta^b \eta^c, \quad (40)$$

from (39) we get that

$$\alpha_2 = -f_{abc} \eta^{*\mu} \eta^a \eta^b A_\mu^c, \quad (41)$$

hence

$$\delta\alpha_3 + \gamma\alpha_2 = \partial_\mu (-f_{abc} \eta^{*\mu} \eta^a \eta^b \eta^c). \quad (42)$$

By projecting (29) on antighost number one, we infer

$$\delta\alpha_2 + \gamma\alpha_1 = \partial_\mu w^\mu. \quad (43)$$

Starting from

$$\delta\alpha_2 = 2f_{abc} (\partial_\nu B^{*\nu\mu}) \eta^a \eta^b A_\mu^c, \quad (44)$$

we arrive at

$$\alpha_1 = 2B^{*\mu\nu} \eta_a \partial_{[\mu} A_{\nu]}^a, \quad (45)$$

which leads to

$$\delta\alpha_2 + \gamma\alpha_1 = \partial_\mu (2f_{abc} B^{*\mu\nu} \eta^a \eta^b A_\nu^c). \quad (46)$$

With α_1 at hand, we proceed to determine α_0 as solution to the equation

$$\delta\alpha_1 + \gamma\alpha_0 = \partial_\mu w^\mu. \quad (47)$$

On account of (45), we have

$$\delta\alpha_1 = (\partial_\rho F^{\rho\mu\nu}) \eta_a \partial_{[\mu} A_{\nu]}^a. \quad (48)$$

Then,

$$\alpha_0 = \frac{1}{3} F^{\mu\nu\rho} \left(f_{abc} A_\mu^a A_\nu^b A_\rho^c + A_\mu^a F_{a\nu\rho} + A_\rho^a F_{a\mu\nu} + A_\nu^a F_{a\rho\mu} \right), \quad (49)$$

such that

$$\delta\alpha_1 + \gamma\alpha_0 = \partial_\mu \left(F^{\mu\nu\rho} \eta_a \partial_{[\nu} A_{\rho]}^a \right). \quad (50)$$

In this way, we have generated the first-order deformation under the form

$$S_1 = \int d^D x \left(\frac{1}{3} F^{\mu\nu\rho} \left(f_{abc} A_\mu^a A_\nu^b A_\rho^c + A_\mu^a F_{a\nu\rho} + A_\rho^a F_{a\mu\nu} + A_\nu^a F_{a\rho\mu} \right) + \right. \\ \left. 2B^{*\mu\nu} \eta_a \partial_{[\mu} A_{\nu]}^a - f_{abc} \eta^{*\mu} \eta^a \eta^b A_\mu^c + \frac{1}{3} f_{abc} \eta^{*\mu} \eta^a \eta^b \eta^c \right). \quad (51)$$

The existence of α is equivalent to the consistency of the interaction up to order g . The interaction is then consistent also to order g^2 if and only if ω is s -exact modulo d (see (30)). By direct computation, we deduce

$$(S_1, S_1) = \frac{4}{3} \int d^D x \left(f_{abc} A_\mu^a A_\nu^b A_\rho^c + A_\mu^a F_{a\nu\rho} + A_\rho^a F_{a\mu\nu} + \right. \\ \left. A_\nu^a F_{a\rho\mu} \right) \left((\partial^\mu \eta_d) \partial^{[\nu} A^{\rho]d} + (\partial^\rho \eta_d) \partial^{[\mu} A^{\nu]d} + (\partial^\nu \eta_d) \partial^{[\rho} A^{\mu]d} \right) \\ = \int d^D x \omega. \quad (52)$$

From the obvious relations

$$s \left(f_{abc} A_\mu^a A_\nu^b A_\rho^c + A_\mu^a F_{a\nu\rho} + A_\rho^a F_{a\mu\nu} + A_\nu^a F_{a\rho\mu} \right) = \\ (\partial_\mu \eta_d) \partial_{[\nu} A_{\rho]}^d + (\partial_\rho \eta_d) \partial_{[\mu} A_{\nu]}^d + (\partial_\nu \eta_d) \partial_{[\rho} A_{\mu]}^d, \quad (53)$$

we derive that

$$\omega = s \left(\frac{2}{3} \left(f_{abc} A_\mu^a A_\nu^b A_\rho^c + A_\mu^a F_{a\nu\rho} + A_\rho^a F_{a\mu\nu} + A_\nu^a F_{a\rho\mu} \right)^2 \right), \quad (54)$$

which further yields

$$S_2 = -\frac{1}{3} \int d^D x \left(f_{abc} A_\mu^a A_\nu^b A_\rho^c + A_\mu^a F_{a\nu\rho} + A_\rho^a F_{a\mu\nu} + A_\nu^a F_{a\rho\mu} \right)^2. \quad (55)$$

The higher-order equations are then satisfied with

$$S_3 = S_4 = \dots = 0. \quad (56)$$

In this way, we inferred that

$$\begin{aligned}\bar{S} = \int d^D x \Big(& -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} - \frac{1}{12}H_{\mu\nu\rho} H^{\mu\nu\rho} + A_a^{*\mu} (D_\mu)^a{}_b \eta^b - \\ & \frac{1}{2}f_{bc}^a \eta_a^* \eta^b \eta^c + B^{*\mu\nu} \left(\partial_{[\mu} \eta_{\nu]} + 2g\eta_a \partial_{[\mu} A_{\nu]}^a \right) + \eta^{*\mu} \partial_\mu \eta - \\ & g f_{abc} \eta^{*\mu} \eta^a \eta^b A_\mu^c + \frac{g}{3} f_{abc} \eta^* \eta^a \eta^b \eta^c \Big),\end{aligned}\quad (57)$$

is solution to the master equation (27) of our deformation problem. Clearly, solution (57) is covariant and local. The antifield-independent piece in (57)

$$\bar{S}_0 = \int d^D x \left(-\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} - \frac{1}{12}H_{\mu\nu\rho} H^{\mu\nu\rho} \right), \quad (58)$$

with

$$H_{\mu\nu\rho} = F_{\mu\nu\rho} - 2g \left(f_{abc} A_\mu^a A_\nu^b A_\rho^c + A_{[\mu}^a F_{a\nu\rho]} \right), \quad (59)$$

describes the combined Yang-Mills-2-form system coupled through the Yang-Mills Chern-Simons term, known as the Chapline-Manton model. From the terms linear in the antifields of the original fields we find that the gauge transformations of the Yang-Mills vector potential are not deformed, while those of the 2-form are so, namely,

$$\bar{\delta}_\epsilon B_{\mu\nu} = \partial_{[\mu} \epsilon_{\nu]} + 2g\epsilon_a \partial_{[\mu} A_{\nu]}^a. \quad (60)$$

The analysis of the terms linear in the antifields of the ghosts emphasizes that the commutators among the gauge generators of the Yang-Mills fields remain unaffected, but there appear non-vanishing commutators between the gauge generators of the 2-form and Yang-Mills fields (see the supplementary term $-g f_{abc} \eta^{*\mu} \eta^a \eta^b A_\mu^c$). Thus, the deformation problem analyzed here maintains the covariance and space-time locality, and leads to the deformation of both gauge transformations and gauge algebra. This completes our approach.

4 Conclusion

To conclude with, in this paper we have investigated the consistent interaction that can be introduced between the Yang-Mills vector potential and a 2-form. Starting with the BRST differential for the “free” theory, $s = \delta + \gamma$,

we initially compute $H^0(s|d)$ by expanding the co-cycles accordingly the antighost number, and generate in this way the consistent first-order deformation. Next, we prove that the deformation is also second-order consistent and, moreover, matches the higher-order deformation equations. As a result, we are led precisely to the Chern-Simons couplings of a 2-form to Yang-Mills gauge fields, that imply the deformation of both gauge transformations and their algebra. Finally, we remark that in the case of the coupled model the ghost for ghost η and the primitive form $f_{abc}\eta^a\eta^b\eta^c$ do not enter the cohomology since $f_{abc}\eta^a\eta^b\eta^c = \bar{s}\left(\frac{3}{g}\eta\right)$, which indicates that the primitive form is \bar{s} -exact and η is no longer closed. In turn, this last property underlies the Green-Schwarz anomaly cancellation mechanism.

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